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Localisability in classical mechanics

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Abstract. A reasonable definition for the notion of localisability in classical mechanics is given. It could explain in a satisfactory way the properties of the relativistic photon and the fact that some relativistic systems predicted by Poincaré invariance, such as tachyons, do not appear in Nature.

1. Introduction

In this paper we analyse the notion of localisability for classical systems, described in the framework of Hamiltonian formalism.

The notion of localisability for elementary relativistic systems has appeared in the framework of quantum mechanics [1, 2]. The idea is the following. It is assumed from the beginning that the elementary system is pointlike, so the configuration space is identical with the 'physical' space $\mathbb{R}^3$. Then, according to Wightman [2], the position observable is a projection valued measure based on $\mathbb{R}^3$. If one takes into account the relativistic invariance with respect to the Euclidean group SE(3), then it follows from, natural compatiblity conditions, that the mathematical objects describing localisability in quantum mechanics are the systems of imprimitivity (which can be analysed using Mackey's theory). For recent developments see [3].

Here, we propose a natural definition for the notion of localisability in classical mechanics, using the framework of the Hamiltonian formalism. In fact, the basic motivation of the Hamiltonian formalism reposes on the observation that for a physical system with the configuration space Q, one has a natural symplectic manifold $M \equiv T^*(Q)$—the phase space—and the evolution is governed by a Hamiltonian vector field (see [4], the preview). One can say that such a system is localisable on Q: the 'position' of the system in the state $(q, p) \in T^*(Q)$ is $q \in Q$. (Of course, it is implicitly assumed that one has a prescription by which one determines the point $q \in Q$, by measurements performed in the 'physical' space $\mathbb{R}^3$.)

On the other hand, one can abandon completely the requirement that the phase space M is a cotangent bundle to some configuration manifold Q, and one admits that the phase space can be any symplectic manifold. In this approach, an elementary relativistic system is, by definition, any homogeneous symplectic manifold for the symmetry group of the problem [5-9] and [10, the footnote of p 180].

Our definition of localisability in the Hamiltonian formalism could be characterised as an attempt to preserve the original motivation for the introduction of the notion of phase space (an explained above). Loosely speaking, we propose that a reasonable phase space M must be fibred over a phase space of the type $T^*(Q)$; then the projection

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on $T^*(Q)$ describes the configuration of the system (and the conjugated momenta), and the fibres describe the internal degrees of freedom, e.g. the spin. If the Euclidean group SE(3) acts on Q and on M, we have a natural compatibility condition which has a clear counterpart in the quantum mechanics analysis (compare axiom V p 848 of [2] with definition 1 in section 2 of this paper).

These conditions are of kinematical nature. We think that they must be supplemented by another condition of dynamical nature. Suppose that the Hamiltonian H of the system 'depends only on the variables $(q,p) \in T^*(Q)$' (more precisely H factorises to $T^*(Q)$). Then one can determine the time evolution of $(q,p) \in T^*(Q)$ in two ways: working with the Hamiltonian H on the phase space M and projecting the integral curves of H on $T^*(Q)$, or working directly on $T^*(Q)$ with the factorised Hamiltonian. We think that it is reasonable to suppose that the two alternatives give the same result. We call this new condition strict localisability and we will admit that any physical system must be strictly localisable on some configuration space Q.

It remains to choose the configuration space Q. One could take $Q = \mathbb{R}^3$ as in the analysis of Wightman; this possibility was explored previously [11]. We think that the identification of the configuration space Q with the 'physical' space $\mathbb{R}^3$ is too restrictive. In fact, from the very beginning of the analytical mechanics it was admitted that these two objects can be different.

Here we propose as admissible configuration spaces all Euclidean homogeneous manifolds. This hypothesis agrees with a previous suggestion [12, p 584] made in the context of quantum mechanics.

From the physical point of view the results which follow from these hypotheses are interesting. Although there are many Euclidean homogeneous manifolds, only two of them can be configuration spaces (in the strict sense) for the elementary relativistic systems with respect to the Poincaré group: $\mathbb{R}^3$ and SE(3)/SE(2). The first one can be a configuration space for non-zero mass systems (as in the analysis of Wightman). The second one can be a configuration space for a certain zero mass system; this agrees with the proposal from [12].

The physical interpretation of the configuration space SE(3)/SE(2) is not completely clear. A possible interpretation is based on a particular realisation of this homogeneous space as the manifold of bidimensional (oriented) planes in $\mathbb{R}^3$; then a system localised on such a configuration space could be imagined as a bidimensional object, e.g. a plane wave. Maybe this interpretation of the configuration space SE(3)/SE(2) is not completely satisfactory, and a better one can be found. In any case, the main physical significance of our result is the following: some hypothetical particles such as tachyons or particles of zero mass and infinite spin are not strictly localisable (according to our definition). This can perhaps explain why they are not found in Nature.

To summarise, we propose to define an elementary relativistic system (for the Galilei or Poincaré group) as a homogeneous symplectic manifold M for the corresponding group, together with a configuration space Q. Also M and Q are connected by some natural compatibility conditions which have been presented above, and will be formulated in a rigorous fashion in section 2. Some necessary conditions for (strict) localisability, which will be needed are also given in section 2. In section 3 we give a complete analysis of this problem for the Poincaré group. The analysis reduces to the tedious computation of some Poisson brackets, and to the integration of some nonlinear systems with partial differential equations. It is interesting to note that all the cases can be integrated completely. This seems to indicate that the results can be obtained in a more abstract fashion. We mention also that in the course of the proof we re-obtain in a more natural way a number of formulae which have already appeared in the literature [13-15]. Besides new formulae connected with the new configuration space SE(3)/SE(2), we think that our point of view has the merit of stressing that the underlying structure for all these formulae is that of (strict) localisability.

Let us comment on the connection between our results and other approaches in the literature. Our notion of localisability is closely related in spirit with the work of [7] (see also [8] and [9]) where one supposes essentially that for every point from the phase space one has a line of universe i.e. a line in $\mathbb{R}^4$. More precisely one requires that there exists an equivariant map from M onto some manifold of lines from $\mathbb{R}^4$.

Another related notion of localisability appears in [16]. Finally, we mention the approach of Souriau [10], based on the notion of evolution space, which in a certain sense generalises the notion of configuration space including the time on equal footing.

From the physical point of view our result concerning the localisability of the zero mass systems, imagined as planes in $\mathbb{R}^3$, is closely related with the result of Souriau [10, see footnote p 191]. The same physical idea appears also in [17] in the framework of a quantum analysis but is exploited differently.

Some final comments are made in section 4.

2. The notion of localisability in Hamiltonian formalism

2.1. By definition, the Euclidean group E(3) is:

$$\mathrm{E}(3)=\{(R, \boldsymbol{a})|(R\boldsymbol{x}, R\boldsymbol{y})_{\mathbb{R}^3}=(\boldsymbol{x}, \boldsymbol{y})_{\mathbb{R}^3}\ \forall \boldsymbol{x}, \boldsymbol{y}\in\mathbb{R}^3,\ \boldsymbol{a}\in\mathbb{R}^3\}$$

with the composition law:

$$(R, \boldsymbol{a})(R', \boldsymbol{a}')=(RR', \boldsymbol{a}+R\boldsymbol{a}').$$

Here

$$(\boldsymbol{x}, \boldsymbol{y})_{\mathbb{R}^3}=x_1y_1+x_2y_2+x_3y_3 \qquad (\|\boldsymbol{x}\|_{\mathbb{R}^3})^2\equiv(\boldsymbol{x}, \boldsymbol{x})_{\mathbb{R}^3}.$$

Also, the special Euclidean group SE(3) is:

$$\mathrm{SE}(3)\equiv\{(\boldsymbol{R}, \boldsymbol{a})\in \mathrm{E}(3)|\det R=1\}.$$

Here det R is defined by $Re_1\wedge Re_2\wedge Re_3=(\det R)e_1\wedge e_2\wedge e_3$ where e_1, e_2, e_3 is the canonical basis in $\mathbb{R}^3$.

2.2.

Definition 1. Let (M, Ω) a symplectic manifold on which the group SE(3) acts symplectically, and let Q be a SE(3)-homogeneous connected manifold. We say that the system (M, Ω) is localisable on the configuration space Q if there exists an SE(3) morphism φ: $M\to T^*(Q)$, where on $T^*(Q)$ one considers the natural lifted action of SE(3) (see [4, p 283]). Two configuration spaces Q_1 and Q_2 are considered identical if they are SE(3) diffeomorphic.

Remarks. (i) This definition is very natural: if Q is a configuration space, then the usual prescription for the canonical formalism is to take as phase space the cotangent bundle $T^*(Q)$. Definition 1 is more general in the sense that it admits a system with internal structure.

(ii) If we want to consider the spatial inversion, then we must substitute in this definition E(3) instead of SE(3).

(iii) Suppose Q and $\hat{Q}$ are two configuration spaces such that Q covers $\hat{Q}$ i.e. there exists a SE(3) morphism $\psi : Q \to \hat{Q}$. Then we have a natural SE(3) morphism $\tilde{\psi} : T^*(Q) \to T^*(\hat{Q})$. It follows that if the system (M, Ω) is localisable on Q, then it is also localisable on $\hat{Q}$: if φ: $M \to T^*(Q)$ is a SE(3) morphism, then $\tilde{\psi} \circ \varphi : M \to T^*(\hat{Q})$ is also a SE(3) morphism.

Definition 2. Let (M, Ω) be a system localised on Q, and let $\pi : T^*(Q) \to Q$ be the canonical projection. Then the Q-valued observable $X \equiv \pi \circ \varphi$ is called the position (or configuration) of the system.

If (M, Ω) is a symplectic manifold and $H \in \mathcal{F}(M)$ we denote by X_H the Hamiltonian vector field associated with H; by definition:

$$i_{X_H} \Omega = \mathrm{d}H.$$

In the canonical formalism, the evolutions are integral curves of X_H.

Definition 3. Let (M_i, Ω_i), $i = 1, 2$ be two symplectic manifolds and $\varphi : M_1 \to M_2$ a smooth map. Let $h \in \mathcal{F}(M_2)$ and $x_1 : \mathbb{R} \to M_1$, an integral curve of $X_{h \circ \varphi}$. If $x_2 = \varphi \circ x_1$ is an integral curve for X_h, for all curves x_1, then we say that the two symplectic structures are h-compatible.

Definition 4. Let (M_i, Ω_i), $i = 1, 2$ be two symplectic manifolds and $\varphi : M_1 \to M_2$ a smooth map. If the two symplectic structures are h-compatible for any $h \in \mathcal{F}(M_2)$, then we say that φ is natural.

Definition 5. Let (M, Ω) be a symplectic manifold such that the corresponding system is localisable on the manifold Q. We say that the system is strictly localisable if the map $\varphi : M \to T^*(Q)$ from definition 1 is natural.

Definition 6. Let G be a Lie group such that $\mathrm{SE}(3) \subset G$. An elementary relativistic system for G is a triplet (M, Ω, Q) where (M, Ω) is a symplectic homogeneous manifold for G, Q is a SE(3) homogeneous and connected manifolds and the corresponding system is strictly localisable on Q.

In the following we give a simple criterion to decide if a system associated with a symplectic manifold (M, Ω) is localisable on the manifold Q.

Proposition. Let M_1 and M_2 be two G-spaces and $M_i = \bigcup_{\alpha \in A_i} \vartheta_{i\alpha}$ $(i = 1, 2)$ be the decomposition of M_i into G-orbits. Let $\vartheta_{i\alpha} \simeq G/H_{i\alpha}$ ($H_{i\alpha} \subset G$ are closed subgroups). Suppose that $\varphi : M_1 \to M_2$ is a G-morphism. Then for any $\alpha \in A_1$, there exists $\beta \in A_2$ such that $H_{1\alpha}$ is included, up to conjugacy, in $H_{2\beta}$, i.e. there exists $g \in G$ such that $gH_{1\alpha}g^{-1} \subset H_{2\beta}$.

The proof is trivial. We will consider the case $G = \mathrm{SE}(3)$, $M_1 = M$ (the symplectic manifold) and $M_2 = T^*(Q)$.

2.3. From section 2.2 it follows that we must first find all SE(3) orbits from $T^*(Q)$ for all SE(3) homogeneous connected manifolds Q. We denote by Lie G, the Lie algebra of the Lie group G. Then we have the following proposition.

Proposition 1. Let $Q \simeq \mathrm{SE}(3)/K$ ($K \subset \mathrm{SE}(3)$ is a closed subgroup). Then, the SE(3) orbits from $T^*(Q)$ are of the type $\mathrm{SE}(3)/G_{(K,\lambda)}$, where $\lambda \in (\mathrm{Lie\ SE}(3)/\mathrm{Lie}\ K)^*$ and

$$\mathrm{Lie}\ G_{(K,\lambda)} = \{\xi \in \mathrm{Lie}\ K \mid \lambda([\xi, \eta]) = 0, \forall \eta \in \mathrm{Lie}\ K\}.$$

Proof. We identify $T^*(\mathrm{SE}(3)/K)_K \simeq (\mathrm{Lie\ SE}(3)/\mathrm{Lie}\ K)^* \hookrightarrow (\mathrm{Lie\ SE}(3))^*$ (the last inclusion is non-canonical). Let $(K, \lambda) \in T^*(\mathrm{SE}(3)/K)_K$. Then we have

$$\begin{aligned}\mathrm{Lie}\ G_{(K,\lambda)} &= \{\xi \in \mathrm{Lie}\ K \mid \mathrm{Ad}_{\exp \xi}\lambda = \lambda\}\\ &= \{\xi \in \mathrm{Lie}\ K \mid \lambda(\mathrm{Ad}_{\exp(-t\xi)}\eta) = \lambda(\eta), \forall \eta \in \mathrm{Lie}\ K\}\\ &= \{\xi \in \mathrm{Lie}\ K \mid \lambda(\mathrm{ad}(\xi)\eta) = 0, \forall \eta \in \mathrm{Lie}\ K\}.\end{aligned}$$

To do practical computations it is convenient to identify, as in [18], $\mathrm{Lie\ SE}(3) \simeq \Lambda^2\mathbb{R}^3 + \mathbb{R}^3$ with the Lie bracket:

$$[(\alpha, \boldsymbol{x}), (\beta, \boldsymbol{y})] = ([\alpha, \beta], A_\alpha \boldsymbol{y} - A_\beta \boldsymbol{x}).$$

Here $A_\alpha \in \mathrm{End}\ (\mathbb{R}^3)$ is defined by:

$$A_{\mu \wedge v}\boldsymbol{x} \equiv \boldsymbol{\mu}(\boldsymbol{v}, \boldsymbol{x})_{\mathbb{R}^3} - (\boldsymbol{\mu}, \boldsymbol{x})_{\mathbb{R}^3}\boldsymbol{v}$$

and linearity.

Then we have the following.

Proposition 2. Let $(\alpha, \boldsymbol{x}) \in \mathrm{Lie\ SE}(3)$. Then:

$$\mathrm{Lie}\ G_{(K,(\alpha,x))} \simeq \{(\beta, \boldsymbol{y}) \in \mathrm{Lie}\ K \mid [\alpha, \beta] - \boldsymbol{x} \wedge \boldsymbol{y} = 0, A_\beta \boldsymbol{x} = 0\}.$$

Proof. We identify $(\mathrm{Lie\ SE}(3))^* \simeq \mathrm{Lie\ SE}(3)$ with the scalar product in $\Lambda^2\mathbb{R}^3 + \mathbb{R}^3$ induced by the scalar product in $\mathbb{R}^3$. Then the result follows by a simple computation from proposition 1, and the expression of the Lie bracket given above.

Finally, we apply this proposition for all SE(3)-homogeneous connected manifolds $Q = \mathrm{SE}(3)/K$. Because of remark (iii) above, we have to consider only the set of 'maximal' SE(3) manifolds i.e. a set $\mathscr{C}$ of SE(3) manifolds such that any SE(3) manifold is covered by a SE(3) manifold from $\mathscr{C}$. Such a list of manifolds $\mathscr{C}$ can be determined by Lie algebraic methods (see e.g. [19]). They correspond to the following subgroups K:

Case (1) $\{(\mathbb{1}, \zeta e_3) \mid \zeta \in \mathbb{R}\}$.

Case (2_μ) $\{(R(e_3, \varphi), \mu\varphi e_3) \mid \varphi \in [0, 2\pi)\}$ $\mu \in \mathbb{R}$

(here $R(\boldsymbol{\nu}, \varphi)$ is the rotation of angle φ around $\boldsymbol{\nu} \in S^2$).

Case (3) $\{(R(e_3, \varphi), \zeta e_3) \mid \varphi \in [0, 2\pi), \zeta \in \mathbb{R}\}$.

Case (4) $\{(\mathbb{1}, \boldsymbol{a}) \mid a_3 = 0\}$.

Case (5) $\{(R, \boldsymbol{0}) \mid R \in \mathrm{SO}(3)\}$.

Case (6_μ) $\{(R(e_3, \varphi), \boldsymbol{a}) \mid \varphi \in [0, 2\pi), a_3 = \mu\varphi\}$ $\mu \in \mathbb{R}$.

Case (7) $\{(R(e_3, \varphi), \boldsymbol{a}) \mid \varphi \in [0, 2\pi)\}$.

Case (8) $\{(\mathbb{1}, \boldsymbol{a})\}$.

Case (9) $\{(\mathbb{1}, \boldsymbol{0})\}$.

Case (10) SE(3).

We will use systematically in the following this system of indexation of the elements of $\mathscr{C}$ (and of the corresponding groups K).

By a direct application of proposition 2 we get proposition 3.

Proposition 3. The SE(3) orbits from $T^*(\mathrm{SE}(3)/K)$ are of the type $\mathrm{SE}(3)/N$, where N can be:

Case (1), (2), (4)	$\{(\mathbb{1}, \mathbf{0})\}$	and K
Case (3)	$\{(\mathbb{1}, \mathbf{0})\}, \{(\mathbb{1}, \zeta e_3) \mid \zeta \in \mathbb{R}\}$	and K
Case (5)	$\{(R(e_3, \varphi), \mathbf{0}) \mid \varphi \in [0, 2\pi)\}$	and K
Case (6_μ)	$\{(\mathbb{1}, \boldsymbol{a}) \mid a_3 = 0\}, \{(R(e_3, \varphi), \varphi\mu e_3) \mid \varphi \in [0, 2\pi)\}$	and K
Case (7)	$\{(\mathbb{1}, \boldsymbol{a})\}$	and K
Case (8)-(10)	K.	

This proposition will enable us to rule out the existence of a SE(3) morphism $\varphi : M \to T^*(Q)$ in many cases.

2.4. We want to characterise here more conveniently the condition of strict localisability. We have the following slight generalisation of the Jacobi theorem [4, p 194].

Proposition. Let (M_i, Ω_i) $(i = 1, 2)$ two symplectic manifolds and $\varphi : M_1 \to M_2$ a smooth map. Then φ is natural iff for any $f, g \in \mathscr{F}(M_2)$:

$$\{f \circ \varphi, g \circ \varphi\}_{M_1} = \{f, g\}_{M_2} \circ \varphi. \tag{2.1}$$

Here $\{\ ,\ \}_{M_i}$ is the Poisson bracket on M_i.

The proof is fairly well known and elementary, so we omit it.

Remark (i) Taking into account this proposition, the definition of localisability given here becomes a generalisation of a similar definition from a preceding paper [11].

(ii) Suppose that the manifold Q covers the manifold $\hat{Q}$ and that the system (M, Ω) is strictly localisable on $\hat{Q}$. Because the map $\tilde{\psi} : T^*(Q) \to T^*(\hat{Q})$ constructed in remark (iii) from section 2.2 is symplectic, the system (M, Ω) is localised on Q also. Combined with remark (iii) from section 2.2 this enables us to consider first only the configuration spaces indicated at 3 and then, in case of an affirmative answer, to classify the manifolds which can be covered by Q.

2.5. To verify the condition of strict localisability (2.1) we need some convenient realisation for some SE(3) homogeneous spaces Q, and for $T^*(Q)$ (see the list before proposition 3).

Case (2_0) $\mathrm{SE}(3)/\{(R(e_3, \varphi), \mathbf{0})\} \simeq S^2 \times \mathbb{R}^3$ with the action:

$$(R, \boldsymbol{a})(\boldsymbol{\nu}, \boldsymbol{q}) = (R\boldsymbol{\nu}, R\boldsymbol{q} + \boldsymbol{a}).$$

We identify $T_{(\boldsymbol{\nu}, \boldsymbol{q})}(Q) \simeq \{(\boldsymbol{\mu}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0\}$ by:

$$(\boldsymbol{\mu}, \boldsymbol{v})_{(\boldsymbol{\nu}, \boldsymbol{q})} f \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{\nu} + s\boldsymbol{\mu} + o(s^2), \boldsymbol{q} + s\boldsymbol{v})|_{s=0}$$

for any $f \in \mathscr{F}(S^2 \times \mathbb{R}^3)$. Then we identify $T^*_{(\boldsymbol{\nu}, \boldsymbol{q})}(Q) \simeq T_{(\boldsymbol{\nu}, \boldsymbol{q})}(Q)$ with the bilinear form:

$$\langle(\boldsymbol{\mu}, \boldsymbol{v}), (\boldsymbol{\mu}', \boldsymbol{v}')\rangle = (\boldsymbol{\mu}, \boldsymbol{\mu}')_{\mathbb{R}^3} + (\boldsymbol{v}, \boldsymbol{v}')_{\mathbb{R}^3}$$

which is non-degenerate as can easily be shown.

Then the lifted action of SE(3) is:

$$\tilde{\varphi}_{R, \boldsymbol{a}}(\boldsymbol{\nu}, \boldsymbol{q}, \boldsymbol{\mu}, \boldsymbol{p}) = (R\boldsymbol{\nu}, R\boldsymbol{q} + \boldsymbol{a}, R\boldsymbol{\mu}, R\boldsymbol{p}).$$

Case (3) $\mathrm{SE}(3)/\{(R(e_3, \varphi), \zeta e_3)\} \simeq \{(\boldsymbol{\nu}, \boldsymbol{q}) \in S^2 \times \mathbb{R}^3 | (\boldsymbol{\nu}, \boldsymbol{q})_{\mathbb{R}^3} = 0\} \equiv Q_0$ with the action

$$(R, \boldsymbol{a})(\boldsymbol{\nu}, \boldsymbol{q}) = (R\boldsymbol{\nu}, R\boldsymbol{q} + P'_{R\nu}\boldsymbol{a}).$$

Here P_ν is the orthogonal projection on the linear subspace generated by $\boldsymbol{\nu}$, and $P'_\nu = 1 - P_\nu$.

We identify then

$$T_{\nu,q}(Q_0) \simeq \{(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^3 \times \mathbb{R}^3 | (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0, (\boldsymbol{\nu}, \boldsymbol{v})_{\mathbb{R}^3} + (\boldsymbol{\mu}, \boldsymbol{q})_{\mathbb{R}^3} = 0\}$$

by

$$(\boldsymbol{\mu}, \boldsymbol{v})_{(\nu,q)} f \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{\nu} + s\boldsymbol{\mu} + o(s^2), \boldsymbol{q} + s\boldsymbol{v} + o(s^2))|_{s=0}.$$

Then as in case (2_0) we identify $T^*_{v,q}(Q_0) \simeq T_{\nu,q}(Q_0)$ and we get, for the lifted action of SE(3),

$$\tilde{\varphi}_{R,a}(\boldsymbol{\nu}, \boldsymbol{q}, \boldsymbol{\mu}, \boldsymbol{p}) = (R\boldsymbol{\nu}, R\boldsymbol{q} + P'_{R\nu}\boldsymbol{a}, R\boldsymbol{\mu} + P'_{R\nu} P_{\langle a, Rp\rangle}\boldsymbol{\nu} + \beta(R\boldsymbol{q} + P'_{R\nu}\boldsymbol{a}), R\boldsymbol{p} + \beta R\boldsymbol{\nu}).$$

Here

$$\beta \equiv -[1 + \|R\boldsymbol{q} + P'_{R\nu}\boldsymbol{a}\|^2]^{-1}[(R\boldsymbol{\mu}, \boldsymbol{a})_{\mathbb{R}^3} + (P_{\langle a, Rp\rangle}\boldsymbol{\nu}, R\boldsymbol{q} + P'_{R\nu}\boldsymbol{a})_{\mathbb{R}^3}]$$

and $P_{\langle u,w\rangle}$ is the orthogonal projector on the plane generated by $\boldsymbol{u}$ and $\boldsymbol{w}$

Case (5) $\mathrm{SE}(3)/(R, \mathbf{0}) \simeq \mathbb{R}^3$ with the action:

$$(R, \boldsymbol{q})\boldsymbol{q} = R\boldsymbol{q} + \boldsymbol{a}.$$

We identify $T_q(\mathbb{R}^3) \simeq \mathbb{R}^3$ by:

$$\boldsymbol{v}_q f \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{q} + s\boldsymbol{v})|_{s=0} \qquad \forall f \in \mathscr{F}(\mathbb{R}^3).$$

Then $T^*(\mathbb{R}^3) \simeq \mathbb{R}^3 \times \mathbb{R}^3$ as usual with the lifted action

$$(R, \boldsymbol{a})(\boldsymbol{q}, \boldsymbol{p}) = (R\boldsymbol{q} + \boldsymbol{a}, R\boldsymbol{p}).$$

Case (6_0) $\mathrm{SE}(3)/\{(R(e_3, \varphi), \boldsymbol{a}) | a_3 = 0\} \simeq S^2 \times \mathbb{R}$ with the action:

$$(R, \boldsymbol{a})(\boldsymbol{\nu}, q) = (R\boldsymbol{\nu}, q + (\boldsymbol{a}, R\boldsymbol{\nu})).$$

We identify $T_{(\nu,q)}(S^2 \times \mathbb{R}) \simeq \{(\boldsymbol{\mu}, v) \in \mathbb{R}^3 \times \mathbb{R} | (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0\}$ by

$$(\boldsymbol{\mu}, v)_{(\nu,q)} f \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{\nu} + s\boldsymbol{\mu} + \sigma(s^2), q + sv)|_{s=0}.$$

Then as before

$$T^*(S^2 \times \mathbb{R}) \simeq \{(\boldsymbol{\nu}, q, \boldsymbol{\mu}, p) \in S^2 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} | (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0\}$$

with the lifted action

$$\tilde{\varphi}_{R,a}(\boldsymbol{\nu}, q, \boldsymbol{\mu}, p) = (R\boldsymbol{\nu}, q + (\boldsymbol{a}, R\boldsymbol{\nu})_{\mathbb{R}^3}, R\boldsymbol{\mu} - P'_{R\nu}\boldsymbol{a}, p).$$

Case (7) $SE(3)/\{(R(e_3, \varphi), \boldsymbol{a})\} \simeq S^2$ with the action:

$$(R, \boldsymbol{a})\boldsymbol{\nu} = R\boldsymbol{\nu}.$$

We identify $T_\nu(S^2) \simeq \{\boldsymbol{\mu} \in \mathbb{R}^3 | (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0\}$ by

$$\boldsymbol{\mu}_\nu f \equiv \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{\nu} + s\boldsymbol{\mu} \times \boldsymbol{\nu} + o(s^2))|_{s=0} \qquad \forall f \in \mathscr{F}(s^2).$$

Then we have

$$T^*(S^2) \simeq \{(\boldsymbol{\nu}, \boldsymbol{\mu}) \in S^2 \times \mathbb{R}^3 | (\boldsymbol{\mu}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0\}$$

with the lifted action

$$\tilde{\varphi}_{R,a}(\boldsymbol{\nu}, \boldsymbol{\mu}) = (R\boldsymbol{\nu}, R\boldsymbol{\mu}).$$

2.6. We turn now to the calculation of the Poisson bracket in the cases studied in section 2.5.

Case (5) The following expression is well known

$$\{f, g\} = \sum_{i=1}^{3} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - (f \leftrightarrow g) \tag{2.2}$$

or, equivalently, and more conveniently for our purpose,

$$\{q_i, q_j\} = 0 \tag{2.3}$$

$$\{p_i, p_j\} = 0 \qquad i = \overline{1, 3}. \tag{2.4}$$

$$\{q_i, p_j\} = \delta_{ij} \tag{2.5}$$

Case (7) In the identification for $T^*(S^2)$ from section 2.5 we identify also

$$T_{\nu,\mu}(T^*(S^2)) \simeq \{(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 | (\boldsymbol{a}, \boldsymbol{\nu})_{\mathbb{R}^3} = 0, (\boldsymbol{\nu}, \boldsymbol{b})_{\mathbb{R}^3} + (\boldsymbol{a}, \boldsymbol{\mu})_{\mathbb{R}^3} = 0\}$$

by

$$(\boldsymbol{a}, \boldsymbol{b})_{(\nu,\mu)} f = \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{\nu} + s\boldsymbol{a} + o(s^2), \boldsymbol{\mu} + s\boldsymbol{b} + o(s^2))|_{s=0}$$

and denote for simplicity

$$\frac{\partial}{\partial \nu_i} \equiv (e_i, \boldsymbol{0}), \frac{\partial}{\partial \mu_i} \equiv (\boldsymbol{0}, e_i) \qquad i = \overline{1, 3}.$$

Then the Poisson bracket is:

$$\{f, g\} = \sum_{i,j=1}^{3} (\delta_{ij} - \nu_i \nu_j) \left[\frac{\partial f}{\partial \nu_i} \frac{\partial g}{\partial \mu_j} - (f \leftrightarrow g) \right] - \sum_{i,j=1}^{3} (\nu_i \mu_j - \nu_j \mu_i) \frac{\partial f}{\partial \mu_i} \frac{\partial g}{\partial \mu_j}. \tag{2.6}$$

Proof. In the identifications above, the canonical two form on $T^*(S^2)$ is:

$$\Omega_{\nu,\mu}((\boldsymbol{a}_1, \boldsymbol{b}_1), (\boldsymbol{a}_2, \boldsymbol{b}_2)) = (\boldsymbol{a}_1, \boldsymbol{b}_2)_{\mathbb{R}^3} - (\boldsymbol{a}_2, \boldsymbol{b}_1)_{\mathbb{R}^3}.$$

Formula (2.6) is now a result of a simple computation using the definition of the Poisson bracket: $\{f, g\}_M = -X_f g$.

Equivalently, we have:

$$\{\nu_i, \nu_j\} = 0 \tag{2.7}$$

$$\{\nu_i, \mu_j\} = \delta_{ij} - \nu_i\nu_j \qquad i = \overline{1, 3}. \tag{2.8}$$

$$\{\mu_i, \mu_j\} = \mu_i\nu_j - \mu_j\nu_i \tag{2.9}$$

Case (2_0) The Poisson structure is determined by (2.3)-(2.5), (2.7)-(2.9) and:

$$\{q_i, \nu_j\} = 0 \tag{2.10}$$

$$\{q_i, \mu_j\} = 0 \tag{2.11}$$

$$i = \overline{1, 3}.$$

$$\{p_i, \nu_j\} = 0 \tag{2.12}$$

$$\{p_i, \mu_j\} = 0 \tag{2.13}$$

This can be inferred using the results above.

Case (3) We will need only (2.7)-(2.9) which are also valid here.

Case (6_0) The Poisson structure is determined by (2.7)-(2.9) and

$$\{q, p\} = 1 \tag{2.14}$$

$$\{q, \nu_i\} = 0 \tag{2.15}$$

$$\{q, \mu_i\} = 0 \qquad i = \overline{1, 3}. \tag{2.16}$$

$$\{p, \nu_i\} = 0 \tag{2.17}$$

$$\{p, \mu_i\} = 0 \tag{2.18}$$

2.7. To exploit the condition of G-morphism for the map φ, we will need a simple observation.

If $\boldsymbol{f}$: $\mathbb{R}^3 \to \mathbb{R}^3$ is smooth and verifies

$$\boldsymbol{f}(R\boldsymbol{x}) = R\boldsymbol{f}(\boldsymbol{x}) \qquad \forall R \in \mathrm{SO}(3)$$

we say that $\boldsymbol{f}$ is rotational covariant. It is easy to prove that in this case, $\boldsymbol{f}$ is of the form:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}f(\|\boldsymbol{x}\|^2)_{\mathbb{R}^3}$$

where $f: \mathbb{R}_+ U\{0\} \to \mathbb{R}$ is smooth.

By analogy, if $\boldsymbol{f}$: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is smooth and verifies:

$$\boldsymbol{f}(R\boldsymbol{x}, R\boldsymbol{y}) = R\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$$

we also call it rotational covariant. In this case, $\boldsymbol{f}$ is of the form

$$f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}A + \boldsymbol{y}\boldsymbol{B} + \boldsymbol{x} \times \boldsymbol{y}C \tag{2.19}$$

where A, B and C are smooth functions of $\|\boldsymbol{x}\|^2_{\mathbb{R}^3}$, $\|\boldsymbol{y}\|^2_{\mathbb{R}^3}$ and $(\boldsymbol{x}, \boldsymbol{y})_{\mathbb{R}^3}$. Indeed, for $\boldsymbol{x} \times \boldsymbol{y} \neq 0$, the vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{x} \times \boldsymbol{y}$ are linear independent so we can write:

$$\boldsymbol{f} = \boldsymbol{x}A_1 + \boldsymbol{y}B_1 + \boldsymbol{x} \times \boldsymbol{y}C_1$$

with A_1, B_1, C_1: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ smooth and rotational invariant i.e. $A_1(R\boldsymbol{x}, R\boldsymbol{y}) = A_1(\boldsymbol{x}, \boldsymbol{y})$, etc.

Now it is evident that A_1, B_1 and C_1 are constant on the SO(3) orbits in $\mathbb{R}^3 \times \mathbb{R}^3$ under the natural action. But it is easy to prove that these orbits are indexed by the invariants $\|\boldsymbol{x}\|^2_{\mathbb{R}^3}$, $\|\boldsymbol{y}\|^2_{\mathbb{R}^3}$ and $(\boldsymbol{x}, \boldsymbol{y})_{\mathbb{R}^3}$. So, a formula of type (2.19) follows for $\boldsymbol{x} \times \boldsymbol{y} \neq \boldsymbol{0}$. Because f is smooth, this formula is valid everywhere.

3. Localisability for Poincaré invariant systems

By definition, the Poincaré group is

$$\mathcal{P} \equiv \{(L, a) | L \in \mathrm{End}(\mathbb{R}^4), a \in \mathbb{R}^4, (Lx, Ly)_{\mathbb{R}^4} = (x, y)_{\mathbb{R}^4}, \forall x, y\}$$

with the composition law

$$(L_1, a_1)(L_2, a_2) = (L_1 L_2, a_1 + L_1 a_2).$$

Here $(x, y)_{\mathbb{R}^4} = x_0 y_0 - (\boldsymbol{x}, \boldsymbol{y})_{\mathbb{R}^3}$. We consider here only the proper orthocronous Poincaré group

$$\mathcal{P}^{\uparrow}_{+} \equiv \{(L, a) | \det L > 0, (e_0, Le_0)_{\mathbb{R}^4} > 0\}.$$

Here, e_0, e_1, e_2, e_3 is the canonical basis in $\mathbb{R}^4$ verifying:

$$(e_i, e_j)_{\mathbb{R}^4} \equiv g_{ij} = \begin{cases} 1 & i = j = 0 \\ -1 & i = j = 1, 2, 3 \\ 0 & \text{in other cases} \end{cases}$$

and $\det L$ is defined by:

$$Le_0 \wedge Le_1 \wedge Le_2 \wedge Le_3 = (\det L) e_0 \wedge e_1 \wedge e_2 \wedge e_3 .$$

3.1. As in [18] we identify $(\mathrm{Lie}\ \mathcal{P}^{\uparrow}_{+})^* \simeq \wedge^2\mathbb{R}^4 + \mathbb{R}^4$ with the coadjoint action:

$$\mathrm{Ad}_{L,a}(\Gamma, P) = (L\Gamma + LP \wedge a, LP). \tag{3.1}$$

Then, the homogeneous symplectic manifolds are coadjoint orbits as follows [10], [18]; see also [20]).

(1) $M^{\eta}_{m,s} \equiv \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = m^2, \|\Gamma \wedge P\|^2_{\mathbb{R}^4} = m^2 s^2, \operatorname{sign} P_0 = \eta\}$

$m \in \mathbb{R}_+, s \in \mathbb{R}_+ \cup \{0\}, \eta = \pm 1.$

(2) $M^{\eta}_{s} \equiv \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = 0, \operatorname{sign} P_0 = \eta, *(\Gamma \wedge P) = -sP\}$

$s \in \mathbb{R}, \eta = \pm 1.$

(3) $\tilde{M}^{\eta}_{\rho} \equiv \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = 0, \operatorname{sign} P_0 = \eta, \|\Gamma \wedge P\|^2_{\mathbb{R}^4} = \rho^2\}.$

(4) $M_{m,\rho} \equiv \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = -m^2, \|\Gamma \wedge P\|^2_{\mathbb{R}^4} = m^2 \rho^2\} \qquad m, \rho \in \mathbb{R}_+$

(5) $\tilde{M}^{\eta}_{m,s} \equiv \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = -m^2, \|\Gamma \wedge P\|^2_{\mathbb{R}^4} = -m^2 s^2, \operatorname{sign}(*(\Gamma \wedge P))_0 = -\eta\}$

$m \in \mathbb{R}_+, s \in \mathbb{R}_+ \cup \{0\}, \eta = \pm 1.$

(6) $\tilde{M}_{\lambda_1 \lambda_2} \equiv \{(\Gamma, 0) | \|\Gamma\|^2_{\mathbb{R}^4} = \lambda_1, \frac{1}{2}\Gamma \wedge \Gamma = \lambda_2 e_0 \wedge e_1 \wedge e_2 \wedge e_3\} \qquad \lambda_1, \lambda_2 \in \mathbb{R}$

(7) $M''_{m,0} = \{(\Gamma, P) | \|P\|^2_{\mathbb{R}^4} = -m^2, \Gamma \wedge P = 0\} \qquad m \in \mathbb{R}_+.$

Here $\|P\|^2 \equiv P_0^2 - \|\boldsymbol{P}\|^2_{\mathbb{R}^3}$. It is convenient to revert to three-dimensional notation, by writing Γ uniquely in the form:

$$\Gamma = e_0 \wedge \boldsymbol{K} + (*(e_0 \wedge \boldsymbol{J}))$$

where $\boldsymbol{J}$ and $\boldsymbol{K}$ are three-dimensional vectors. Then the action (3.1), restricted to SE(3), is:

$$\varphi_{R,a}(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (R\boldsymbol{J} + R\boldsymbol{P} \times \boldsymbol{a}, R\boldsymbol{K} + H\boldsymbol{a}, R\boldsymbol{P}, H). \tag{3.2}$$

Here $H \equiv P_0$.

3.2.

Proposition. The possible configuration spaces $Q \in \mathscr{C}$ for the Poincaré invariant systems listed in section 3.1 are among the following:

(1) for $M^{\eta}_{m,0}$: $\mathbb{R}^3$

(2) for $M^{\eta}_{m,s}(s \in \mathbb{R}_+)$, M^{η}_{s} and $\tilde{M}^{\eta}_{\rho}$: $\mathbb{R}^3$, S^2, $S^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}^3$ and Q_0

(3) for $M_{m,\rho}$, $\tilde{M}^{\eta}_{m,s}$ and $M''_{M,0}$: S^2 and Q_0

(4) for $\tilde{M}_{\lambda_1,\lambda_2}$: S^2.

Proof. We decompose first (Lie $\mathscr{P}^{\uparrow}_{+})^*$ in SE(3) orbits. It is easy to establish that there are three types of orbits:

$$(a)\ \{(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) | H = E, \|\boldsymbol{P}\|_{\mathbb{R}^3} = k, (\boldsymbol{J}, \boldsymbol{P})_{\mathbb{R}^3} = \lambda, \|H\boldsymbol{J} + \boldsymbol{K} \times \boldsymbol{P}\|_{\mathbb{R}^3} = k'\} E \in \mathbb{R}^*,$$
$$\lambda \in \mathbb{R},\ k, k' \in \mathbb{R}_+ U\{0\},\ |\lambda E| \leqslant kk'.$$

The stability subgroups are conjugated with (2_0) for $k \in \mathbb{R}_+$, $|\lambda E| < kk'$ or $k = 0$ and $k' \in \mathbb{R}_+$; with (5) for $k = k' = 0$, and with (9) for $k \in \mathbb{R}_+$, $|\lambda E| = kk'$.

$$(b)\ \{(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, 0) | \|\boldsymbol{P}\|_{\mathbb{R}^3} = k, \|\boldsymbol{K}\|_{\mathbb{R}^3} = k', (\boldsymbol{J}, \boldsymbol{P})_{\mathbb{R}^3} = \lambda, (\boldsymbol{P}, \boldsymbol{K})_{\mathbb{R}^3} = \lambda'\}$$
$$k \in \mathbb{R}_+,\ k' \in \mathbb{R}_+ U\{0\},\ \lambda, \lambda' \in \mathbb{R},\ |\lambda'| \leqslant kk'.$$

The stability subgroups are conjugated with (3) for $|\lambda'| < kk'$ and with (1) for $|\lambda'| = kk'$.

$$(c)\ \{(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{0}, 0) | \|\boldsymbol{J}\|_{\mathbb{R}^3} = s, \|\boldsymbol{K}\|_{\mathbb{R}^3} = k, (\boldsymbol{J}, \boldsymbol{K})_{\mathbb{R}^3} = \lambda\} s, k \in \mathbb{R}_+ U\{0\}, \lambda \in \mathbb{R}_+, |\lambda| \leqslant sk.$$

The stability subgroups are conjugated with: (8) for $s \in \mathbb{R}_+$, $|\lambda| < ks$ and with (7) for $s \in \mathbb{R}_+$, $|\lambda| = ks$ or $s = 0$, $k \in \mathbb{R}_+$.

Now we analyse the SE(3)-orbits content of each manifold in section 3.1 and apply section 2.2.

3.3. For the verification of the condition of strict localisability, it will be profitable to work with new coordinates in cases (1) and (3)-(5) in section 3.1. Namely, one introduces the Pauli-Liubanski quadrivector:

$$W = -{}^*(\Gamma \wedge P)$$

or, in three-dimensional notation,

$$W_0 = (\boldsymbol{J}, \boldsymbol{P}) \qquad \boldsymbol{W} = H\boldsymbol{J} + \boldsymbol{K} \times \boldsymbol{P}.$$

Then we have in the new coordinates $\boldsymbol{W}$, $\boldsymbol{K}$, $\boldsymbol{P}$, H:

(1) $M^{\eta}_{m,s} = \{(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) | H^2 - \|\boldsymbol{P}\|^2_{\mathbb{R}^3} = m^2, \text{sign } H = \eta, \lambda^2 - \|\boldsymbol{W}\|^2_{\mathbb{R}^3} = -m^2 s^2\}$

(3) $\tilde{M}^{\eta}_{\rho} = \{(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) | H^2 = \|\boldsymbol{P}\|^2_{\mathbb{R}^3}, \text{sign } H = \eta, \lambda^2 - \|\boldsymbol{W}\|^2_{\mathbb{R}^3} = -\rho^2\}$

(4) $M_{m,\rho} \equiv \{(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) | H^2 - \|\boldsymbol{P}\|^2_{\mathbb{R}^3} = -m^2, \lambda^2 - \|\boldsymbol{W}\|^2_{\mathbb{R}^3} = -m^2 \rho^2\}$

(5) $\tilde{M}^{\eta}_{m,s} \equiv \{(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) | H^2 - \|\boldsymbol{P}\|^2_{\mathbb{R}^3} = -m^2, \lambda^2 - \|\boldsymbol{W}\|^2_{\mathbb{R}^3} = m^2 s^2, \text{sign } \lambda = -\eta\}$.

Here $\lambda \equiv (\boldsymbol{P}, \boldsymbol{W})_{\mathbb{R}^3}/H$. The action of SE(3) in these new coordinates is:

$$\varphi_{R,\boldsymbol{a}}(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (R\boldsymbol{W}, R\boldsymbol{K} + H\boldsymbol{a}, R\boldsymbol{P}, H). \tag{3.3}$$

The elementary Poisson brackets are easy to calculate, using the property of the momentum map

$$\{f_\xi, f_\eta\} = -f_{[\xi, \eta]} \qquad \forall \xi, \eta \in \text{Lie } \mathcal{P}_+^\uparrow$$

and are not given here.

Now, we analyse case by case the possibilities permitted by the proposition in 3.2. For the purpose of illustrating the method, we analyse in detail the first case i.e. $M = M_{m,0}$ and $Q = \mathbb{R}^3$. For the other cases we give only the final results. We emphasise that the computations are long, but straightforward.

3.4. $Q = \mathbb{R}^3$.

3.4.1. The system $M^\eta_{m,0}$ is strictly localisable in $\mathbb{R}^3$.

Proof. (i) It is easy to prove that $\forall (\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) \in M^\eta_{m,0}$, $\exists (R, \boldsymbol{a}) \in \text{SE}(3)$ so that:

$$(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = Ad^\#_{R,\boldsymbol{a}}(\boldsymbol{0}, \boldsymbol{0}, k\boldsymbol{e}_3, H)$$

where $k \equiv \|\boldsymbol{P}\|_{\mathbb{R}^3}$. Using the formula (3.2) we get:

$$\boldsymbol{J} = kR\boldsymbol{e}_3 \times \boldsymbol{a}$$

$$\boldsymbol{K} = H\boldsymbol{a}$$

$$\boldsymbol{P} = kR\boldsymbol{e}_3$$

so we must take $\boldsymbol{a} = \boldsymbol{K}/H$ and $R = R(\boldsymbol{P})$. The morphism φ, if it exists, verifies the following

(*a*) $\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = \varphi(Ad^\#_{R,\boldsymbol{a}}(\boldsymbol{0}, \boldsymbol{0}, k\boldsymbol{e}_3, H)) = (R, \boldsymbol{a}) \cdot \varphi(\boldsymbol{0}, \boldsymbol{0}, k\boldsymbol{e}_3, H)$.

(*b*) It is clear that $\varphi(\boldsymbol{0}, \boldsymbol{0}, k\boldsymbol{e}_3, H) = (\boldsymbol{q}(\boldsymbol{P}), \boldsymbol{p}(\boldsymbol{P}))$

where $\boldsymbol{q}, \boldsymbol{p}: \mathbb{R}^3 \to \mathbb{R}^3$ are smooth. From (*a*) and (*b*) we get $\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H + \boldsymbol{\Lambda}_1(\boldsymbol{P}), \boldsymbol{\Lambda}_2(\boldsymbol{p}))$ where $\boldsymbol{\Lambda}_i: \mathbb{R}^3 \to \mathbb{R}^3$ are smooth $(i = 1, 2)$. The condition of the SE(3) morphism is equivalent with the condition of rotational covariance for the functions Λ_i. So, using section 2.7, we get the most general form of φ:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H + \boldsymbol{P}f(\|\boldsymbol{P}\|^2_{\mathbb{R}^3}), \boldsymbol{P}g(\|\boldsymbol{P}\|^2_{\mathbb{R}^3}). \tag{3.4}$$

This proves that the system $M_{m,0}$ is localisable on $\mathbb{R}^3$.

(ii) We turn now to the question of strict localisability.

Condition (2.5) is equivalent with $g = 1$, and conditions (2.3) and (2.4) are verified identically. So, the most general solution for the problem of strict localisability is:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H + \boldsymbol{P}f(\|\boldsymbol{P}\|^2_{\mathbb{R}^3}, \boldsymbol{P}).$$

Remark. The transformation

$$T^*(\mathbb{R}^3) \ni (\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q} + \boldsymbol{p}f(\|\boldsymbol{p}\|^2_{\mathbb{R}^3}), \boldsymbol{p}) \in T^*(\mathbb{R}^3)$$

is canonical. In the new variables, φ, from above, becomes

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H, \boldsymbol{P}).$$

So, in this case, $\boldsymbol{\varphi}$ is unique, up to canonical transformations on $T^*(Q)$.

3.4.2. The system $M_{m,s}(s \in \mathbb{R}_+)$ is strictly localisable in $\mathbb{R}^3$.

Proof. (i) The most general SE(3) morphism is:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H + \boldsymbol{q}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{p}(\boldsymbol{P}, \boldsymbol{W}))$$

with $\boldsymbol{q}, \boldsymbol{p}$ smooth and rotational covariant. According to section 2.7 this implies:

$$\boldsymbol{p}(\boldsymbol{P}, \boldsymbol{W}) = A(\lambda, H)\boldsymbol{P} + B(\lambda, H)\boldsymbol{W} + C(\lambda, H)\boldsymbol{P} \times \boldsymbol{W}$$

$$\boldsymbol{q}(\boldsymbol{P}, \boldsymbol{W}) = \tilde{A}(\lambda, H)\boldsymbol{P} + \tilde{B}(\lambda, H)\boldsymbol{W} + \tilde{C}(\lambda, H)\boldsymbol{P} \times \boldsymbol{W}.$$

(ii) The condition (2.4) imposes $B = C = 0$. Then (2.5) fixes $A = 1$. So, φ is of the form:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{K}/H + \tilde{A}(\lambda, H)\boldsymbol{P} + \tilde{B}(\lambda, H)\boldsymbol{W} + \tilde{C}(\lambda, H)\boldsymbol{P} \times \boldsymbol{W}, \boldsymbol{P}).$$

Finally, after a long computation (2.3) is shown to be equivalent with

$$m^2H^2\tilde{B}^2 + m^2H^2(H^2 - m^2)\tilde{C}^2 + 2mH\tilde{c} - 1 = 0.$$

and

$$[H(H^2 - m^2)\tilde{C} + 1]\partial\tilde{A}/\partial\lambda = \partial\tilde{B}/\partial H - H^2\tilde{C}(\tilde{B} + \lambda\partial\tilde{B}/\partial\lambda).$$

These equations admit solutions. For $\tilde{B} = 0$, we must take $\tilde{C} = \eta 1/[mH(H + \eta m)]$ and we can take $\tilde{A} = 0$. (This solution alredy appears in the literature [15].) But we can also take $\tilde{C} = 0$, and get the solutions:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = \left(\boldsymbol{K}/H + \frac{\varepsilon}{mH}\left(\boldsymbol{W} - \frac{\lambda}{H}\boldsymbol{P}\right), \boldsymbol{P}\right) \qquad \varepsilon = \pm 1$$

which seems to be new.

Remark. We did not succeed in clarifying the problem of uniqueness, up to a canonical transformation, for φ.

3.4.3. The system M_s^η is localisable in $\mathbb{R}^3$, but is strictly localisable iff $s = 0$.

Proof. (i) The most general SE(3) morphism is also given by (3.4).

(ii) As in section 3.4.1, (2.5) fixes $g = 1$, but (2.3) is true iff $s = 0$.

3.4.4. The system $\tilde{M}_\rho^\eta$ is localisable but not strictly localisable in $\mathbb{R}^3$.

Proof. (i) The most general form of the SE(3) morphism, φ is given by the corresponding formulae from section 3.4.2 (i).

(ii) The condition (2.3) cannot be satisfied (see for some details [11]).

3.5. $Q = S^2 \times \mathbb{R}$

3.5.1. The system M_s^η is localisable on $S^2 \times \mathbb{R}$, and is strictly localisable iff $s = 0$.

Proof. (i) The most general SE(3) morphism is:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = [\varepsilon\boldsymbol{P}/H, \varepsilon(\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^2 + q(H), p(H)[-\boldsymbol{K}/H + \boldsymbol{P}(\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^3], p(H)]$$

with q and p smooth, and $\varepsilon = \pm 1$.

(ii) Equations (2.7), (2.15) and (2.18) are satisfied automatically. (2.8) gives $p(H) = \varepsilon H$, and then (2.14) is also satisfied.

Finally (2.9) and (2.16) are satisfied iff $s = 0$. So, for $s = 0$ we have the most general solution:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\varepsilon\boldsymbol{P}/H, \varepsilon(\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^2 + q(H), \varepsilon[-\boldsymbol{K}/H + \boldsymbol{P}(\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^3], \varepsilon H) \quad (3.5)$$

Remark. By the canonical transform in M_0^η:

$$(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) \mapsto (\boldsymbol{J}, \boldsymbol{K} - \varepsilon \boldsymbol{P} q(H), \boldsymbol{P}, H)$$

we get rid of $q(H)$ in (3.5). Then by the canonical transform in $T^*(S^2 \times \mathbb{R})$;

$$(\boldsymbol{\nu}, q, \boldsymbol{\mu}, p) \mapsto (-\boldsymbol{\nu}, -q, -\boldsymbol{\mu}, -p)$$

we can make $\varepsilon = 1$. So, up to canonical transformations the solution is:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{P}/H, (\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^2, -\boldsymbol{K}/H + \boldsymbol{P}(\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^3, H). \quad (3.6)$$

3.5.2. The system $\tilde{M}_\rho^\eta$ is localisable, but not strictly localisable on $S^2 \times \mathbb{R}$.

Proof. (i) The most general form of the SE(3) morphism φ is

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{\nu}(\boldsymbol{P}, \boldsymbol{W}), q(\lambda, H) + (\boldsymbol{K}, \nu(\boldsymbol{P}, \boldsymbol{W}))/H, \boldsymbol{\mu}(\boldsymbol{P}, \boldsymbol{W})$$
$$- p(\lambda, H) P'_{\nu(\boldsymbol{P}, \boldsymbol{W})} \boldsymbol{K}/H, p(\lambda, H))$$

where $\boldsymbol{\nu}, \boldsymbol{\mu}, q$ and p are smooth, $\|\boldsymbol{\nu}\|_{\mathbb{R}^3} = 1$, $(\boldsymbol{\nu}, \boldsymbol{\mu})_{\mathbb{R}^3} = 0$ and $\boldsymbol{\nu}, \boldsymbol{\mu}$ are rotational covariant. According to section 2.7,

$$\boldsymbol{\nu} = A(\lambda, H)\boldsymbol{P} + B(\lambda, H)\boldsymbol{W} + C(\lambda, H)\boldsymbol{P} \times \boldsymbol{W}$$

$$\boldsymbol{\mu} = \tilde{A}(\lambda, H)\boldsymbol{P} + \tilde{B}(\lambda, H)\boldsymbol{W} + \tilde{C}(\lambda, H)\boldsymbol{P} \times \boldsymbol{W}.$$

(ii) Equation (2.7) has the following solutions:

$$\boldsymbol{\nu} = \varepsilon \boldsymbol{P}/H \qquad \varepsilon = \pm 1 \quad (3.7a)$$

$$\boldsymbol{\nu} = A\boldsymbol{P} + B(H\boldsymbol{W} - \lambda \boldsymbol{P}) + C\boldsymbol{P} \times \boldsymbol{W} \quad (3.7b)$$

where $\partial A/\partial \lambda = 0$ and

$$H^2[\rho^2(B^2 + C^2) + A^2] = 1 \qquad B^2 + C^2 > 0.$$

For (3.7a) we get from (2.8) that $p(H) = \varepsilon H$ as in section 5.1. But (2.9) and (2.16) cannot be satisfied simultaneously. For (3.7b), (2.15) cannot be fulfilled.

3.5.3. The system $M_{m,s}^\eta$ is localisable but not strictly localisable on $S^2 \times \mathbb{R}$.

Proof. Similar to the proof in section 3.5.2.

3.6. $Q = S^2$

3.6.1 The system $\tilde{M}_{\lambda_1, \lambda_2}$ is localisable, on S^2 iff $I_1^2 + I_2^2 > 0$ but not strictly localisable on S^2.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{0}, 0) = (\boldsymbol{\nu}(\boldsymbol{J}, \boldsymbol{K}), \mu(\boldsymbol{J}, \boldsymbol{K}))$$

with $\boldsymbol{\nu}, \boldsymbol{\mu}$ smooth, rotational covariant and verifying $\|\boldsymbol{\nu}\|_{\mathbb{R}^3} = 1$, $(\boldsymbol{\nu}, \boldsymbol{\mu})_{\mathbb{R}^3} = 0$.

(ii) Equation (2.7) cannot be satisfied.

3.6.2. The system $M_{m,p}$ is localisable, but not strictly localisable on S^2.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{\nu}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{\mu}(\boldsymbol{P}, \boldsymbol{W}))$$

with $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ as in section 3.5.2.

(ii) Equation (2.7) implies $\boldsymbol{\nu} = (\varepsilon/(H^2+m^2)^{1/2})\boldsymbol{P}(\varepsilon = \pm 1)$ but (2.8) cannot be satisfied.

3.6.3. The system $\tilde{M}^{\eta}_{m,s}$ is localisable, but not strictly localisable on S^2.

Proof. Identical to the one in section 3.6.2.

3.6.4. The system $M^{\eta}_{m,s}(\in \mathbb{R}_+)$ is localisable, but not strictly localisable on S^2.

Proof. (i) φ is of the same form as in section 3.6.2.

(ii) Equation (2.7) implies $\boldsymbol{\nu} = A\boldsymbol{P}$. But $\|\boldsymbol{\nu}\|^2 = 1$, so we have $\|\boldsymbol{P}\|^2 A^2 = 1$ and we get a contradiction for $\boldsymbol{P} = \boldsymbol{0}$.

3.6.5. The system M^{η}_{s} is localisable, but not strictly localisable on S^2.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\varepsilon \boldsymbol{P}/H, \boldsymbol{0}) \qquad \varepsilon = \pm 1.$$

(ii) Equation (2.8) cannot be fulfilled.

3.6.6. The system $\tilde{M}^{\eta}_{\rho}$ is localisable, but not strictly localisable on S^2.

Proof. (i) φ is of the same form as in section 3.6.2.

(ii) Equation (2.7) has the same solutions as those in section 3.5.2. and (2.8) cannot be fulfilled.

3.6.7. The system $M''_{m,0}$ is localisable, but not strictly localisable on S^2.

Proof. Identical to the one in section 3.6.5.

3.7. $Q = Q_0$

3.7.1. The systems $M_{m,\rho}$, $\tilde{M}^{\eta}_{m,s}$ and $M''_{m,0}$ are not localisable on Q_0.

Proof. For $H \neq 0$, one finds out that φ is of the form:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{\nu}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{x}(\boldsymbol{P}, \boldsymbol{W}) + \boldsymbol{K}/H - (\boldsymbol{K}, \boldsymbol{\nu})\boldsymbol{\nu}/H, \ldots)$$

which cannot be extended smoothly for $H = 0$.

3.7.2. The system M^{η}_{s} is localisable, but not strictly localisable on Q_0.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\varepsilon \boldsymbol{P}/H, \boldsymbol{K}/H - (\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}\boldsymbol{P}/H, \boldsymbol{0}, \boldsymbol{0}) \qquad \varepsilon = \pm 1.$$

(ii) Equation (2.8) cannot be satisfied.

3.7.3. The system $\tilde{M}^{\eta}_{\rho}$ is localisable but not strictly localisable on Q_0.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{\nu}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{x}(\boldsymbol{P}, \boldsymbol{W}) + P'_{\nu(P,W)}\boldsymbol{K}/H, \boldsymbol{0}, \boldsymbol{0})$$

with $\boldsymbol{\nu}$, $\boldsymbol{x}$ smooth, $\|\boldsymbol{\nu}\|_{\mathbb{R}^3} = 1$ and $(\boldsymbol{\nu}, \boldsymbol{x})_{\mathbb{R}^3} = 0$.

(ii) Equation (2.8) cannot be satisfied.

3.8. $Q = S^2 \times \mathbb{R}^3$

3.8.1. The system $M^{\eta}_{m,s}(s \in \mathbb{R}_+)$ is localisable, but not strictly localisable on $S^2 \times \mathbb{R}^3$.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{W}, \boldsymbol{K}, \boldsymbol{P}, H) = (\boldsymbol{\nu}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{x}(\boldsymbol{P}, \boldsymbol{W}) + (P'_{\nu}\boldsymbol{K})/H, \boldsymbol{\mu}(\boldsymbol{P}, \boldsymbol{W}), \boldsymbol{p}(\boldsymbol{P}, \boldsymbol{W}))$$

with $\boldsymbol{\nu}$, $\boldsymbol{x}$, $\boldsymbol{\mu}$, $\boldsymbol{p}$ smooth and rotational covariant $\|\boldsymbol{\nu}\|_{\mathbb{R}^3} = 1$ and $(\boldsymbol{\nu}, \boldsymbol{\mu})_{\mathbb{R}^3} = 0$.

(ii) From (2.7) we get $\boldsymbol{\nu} = A\boldsymbol{P}$ and a contradiction follows as in section 6.4.

3.8.2. The system M^{η}_{s} is localisable but not strictly localisable on $S^2 \times \mathbb{R}^3$.

Proof. (i) φ is of the form:

$$\varphi(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H) = (\varepsilon \boldsymbol{P}/H, x(\|\boldsymbol{P}\|^2_{\mathbb{R}^3})\boldsymbol{P} + P'_{(\varepsilon P/H)}\boldsymbol{K}/H, \boldsymbol{0}, \boldsymbol{P}p(\|\boldsymbol{P}\|^2_{\mathbb{R}^3})$$

with x, p smooth.

(ii) Equation (2.8) is not satisfied.

3.8.3. The system $\tilde{M}^{\eta}_{p}$ is localisable but not strictly localisable on $S^2 \times \mathbb{R}^3$.

Proof. It is very similar to the one in section 3.8.1.

3.9. Collecting these results we have the following.

Theorem. The only elementary relativistic systems for $\mathscr{P}^{\uparrow}_{+}$ are $(M^{\eta}_{m,s}, \Omega, \mathbb{R}^3)$ for $s \in \mathbb{R}_+ U\{0\}$, $(M^{\eta}_{0}, \Omega, \mathbb{R}^3)$, $(M^{\eta}_{0}, \Omega, S^2 \times \mathbb{R})$ and $(M^{\eta}_{0}, \Omega, (S^2 \times \mathbb{R})/\mathbb{Z}_2)$ where the action of $\mathbb{Z}_2$ on $S^2 \times \mathbb{R}$ is:

$$\begin{cases} \mathbb{1} \cdot (\boldsymbol{\nu}, q) = (\boldsymbol{\nu}, q) \\ -\mathbb{1} \cdot (\boldsymbol{\nu}, q) = (-\boldsymbol{\nu}, -q). \end{cases}$$

Proof. It is not very hard to prove that $S^2 \times \mathbb{R}$ can cover only the manifold $(S^2 \times \mathbb{R})/\mathbb{Z}_2$ obtained by the factorisation above.

Remark. It is easy to see that $(S^2 \times \mathbb{R}^3)/\mathbb{Z}_2$ is the manifold of bidimensional planes in $\mathbb{R}^3$. Indeed, one fixes a plane Π in $\mathbb{R}^3$, giving a vector $\boldsymbol{\nu} \in S^2$ which is perpendicular on Π, and a real number q such that $q\boldsymbol{\nu} \in \Pi$; of course one must identify $(\boldsymbol{\nu}, q)$ with $(-\boldsymbol{\nu}, -q)$. It is easy to verify that the action of SE(3) from section 2.5 (6_0) is compatible with this interpretation. Inspecting formula (4.6) we see that $\boldsymbol{\nu} = \boldsymbol{P}/H$ i.e. the plane is perpendicular on the direction of the momentum, and $q = (\boldsymbol{K}, \boldsymbol{P})_{\mathbb{R}^3}/H^2$, i.e. the projection of the centre of motion vector $\boldsymbol{K}/H$ on $\boldsymbol{\nu}$. These observations allow us to interpret Π as the plane wave of a photon. A similar idea was proposed (in the framework of quantum mechanics) in [17], but was exploited differently. What is remarkable is that one does not have to impose this image from outside, it emerges from the notion of strict localisability. Moreover, this is in accordance, in some sense, with an analysis of the same type made by Souriau [10] and based on the notion of evolution space. We also note that $(J, \boldsymbol{K}, \boldsymbol{P}, H)$ and $(\boldsymbol{J}', \boldsymbol{K}', \boldsymbol{P}', H)$ have the same configuration iff $\boldsymbol{P}=+\boldsymbol{P}'$ and $(\boldsymbol{K}-\boldsymbol{K}', \boldsymbol{P})=0$. If $\boldsymbol{K}/H$ is assimilated with the centre of mass of the photon, this would correspond to a translation of the plane along itself which, obviously, does not change the physical situation. The elements of $S^2 \times \mathbb{R}$ can be interpreted as oriented planes in $\mathbb{R}^3$.

3.10. One can include inversions also. We do not give the full analysis, but note only that the system $M_{m,s}(s \in \mathbb{R}_+)$, regarded as a $\mathscr{P}^{\uparrow}$ homogeneous manifold, is strictly localisable (with respect to E(3)) iff the functions $\tilde{B}$ and $\tilde{C}$ from section 3.2 are odd, respectively even, in λ because $\boldsymbol{W}$ is a pseudovector.

4. Final remarks

We have given a reasonable definition for the notion of localisability in classical mechanics in the framework of the Hamiltonian formalism. Then we have analysed from this point of view the homogeneous symplectic manifolds for the Poincaré group. The results have been interesting from two points of view. First, they could explain why some hypothetical particles, predicted by Poincaré invariance only, e.g. the tachyons (i.e. $\tilde{M}_{m,\rho}$ and $\tilde{M}^{\eta}_{m,s}$) or the particles of zero mass and infinite spin (i.e. $\tilde{M}^{\eta}_{\rho}$), do not appear in Nature: they are not strictly localisable. Secondly, we get a new configuration space which could perhaps explain the properties of the classical photon. For this it is necessary to find a reasonable physical interpretation for this new configuration space; a tentative interpretation was made in section 3.

It would be interesting to clarify some connected problems. Firstly, to analyse in the same spirit the notion of evolution space of Souriau. In particular, a 'good' definition for this notion must be such that a particle has an evolution space iff it is strictly localisable. Secondly, of the technical level, it would be desirable to exploit the condition of strict localisability in a less computational way, if possible. It would not be surprising if these two problems can be solved simultaneously.

Finally, the same analysis can be done in quantum mechanics, generalising the work of Wightman. Partial results in this direction have been obtained in [21]. These results corroborate the conclusions of this paper, as regards the localisability of the photon.

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